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Equations of motion in linearised gravity: III Radiation of four-momentum

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Abstract. We suggest a conserved 'energy–momentum tensor' for the Robinson–Trautman solutions of the Einstein and the Einstein–Maxwell vacuum field equations which, in the linear approximation discussed in earlier papers, is shown to yield to first order a rate of radiation of four-momentum equal to the loss of four-momentum by the source. In the case of a charged source we obtain the Larmor formula.

1. Introduction

For the Robinson–Trautman (1962) solutions of the Einstein vacuum field equations the 'main equations', in the terminology of Sachs (1962), can be integrated and the 'subsidiary conditions' reduced to one equation (two equations in the Einstein–Maxwell case). Referring to this one equation as the 'propagation equation', Derry *et al* (1969) point out that 'for these solutions the metric must deterministically evolve in time according to the propagation equation in order that the shear-free property (of the principal null rays) be maintained. Therefore, 'news' in the sense of Bondi, van der Burg and Metzner (1962) is not allowed'. Using asymptotic symmetry linkages introduced by Tamburino and Winicour (1966), Derry *et al* obtain a formula for the energy radiation rate. This radiation rate, if it exists, is in general of *second order* when applied to the linearised form of the Robinson–Trautman fields described in earlier papers (Hogan and Imaeda 1979a, b, c). For a uniformly accelerated source in linearised theory it diverges. In addition, the Larmor formula (cf Synge (1965) p 392), which gives a *first-order* energy radiation rate (in Hogan and Imaeda (1979b) it is assumed that (mass) and $(\text{charge})^2$ are both small of first order) from an accelerating charged mass, might be expected to emerge from a suggested formula for the rate of energy radiated, when applied to the linearised Robinson–Trautman solutions of the Einstein–Maxwell equations.

In this paper we describe a conserved 'energy–momentum tensor' for the Robinson–Trautman solutions. It is motivated partly on mathematical grounds. It is a symmetric, divergence-free, two-index tensor which is invariant under the group of transformations which leave invariant the line element and four-potential under consideration. Furthermore, when it is applied to the linearised Robinson–Trautman fields discussed earlier we find a rate of radiation of four-momentum which is

(1) independent of the time-like three-surface $r=\text{constant}$, across which it is measured, for $0<r<+\infty$;
(2) equal to the rate of loss of four-momentum by the source;
(3) equal to the Larmor energy radiation rate if the source carries charge.

2. Exact theory

Since we shall be interested in both charged and uncharged sources we quote initially the Robinson–Trautman (1962; Robinson (1973) private communication to the author) solutions of the Einstein–Maxwell vacuum field equations. The line element and four-potential are given respectively by

$$ds^2 = 2r^2P^{-2}\,d\zeta\,d\bar{\zeta} - 2\,dr\,d\sigma - h\,d\sigma^2, \tag{2.1}$$

and the one-form

$$\Phi = -F\,d\sigma, \tag{2.2}$$

where

$$P = P(\zeta, \bar{\zeta}, \sigma), \tag{2.3a}$$

$$h = K - 2Hr - 2Mr^{-1} + e^2r^{-2} \qquad (e = e(\sigma)), \tag{2.3b}$$

$$F = e(r^{-1} - w), \tag{2.3c}$$

$$K = \Delta \ln P \qquad (\Delta = 2P^2\,\partial^2/\partial\zeta\,\partial\bar{\zeta}), \tag{2.3d}$$

$$H = \partial(\ln P)/\partial\sigma, \tag{2.3e}$$

$$M = m + 2e^2w \qquad (m = m(\sigma)), \tag{2.3f}$$

$$w = w(\zeta, \bar{\zeta}, \sigma), \tag{2.3g}$$

and the 'subsidiary conditions' referred to in § 1 are

$$\tfrac{1}{4}\Delta K = \dot{M} - 3HM + e^2N, \tag{2.4a}$$

$$\Delta w = e^{-1}\dot{e} - 2H. \tag{2.4b}$$

Here the dot denotes differentiation with respect to σ,

$$N = 2P^2\frac{\partial w}{\partial\zeta}\frac{\partial w}{\partial\bar{\zeta}}, \tag{2.5}$$

and we have chosen units for which $c = G = 1$ and Gaussian electromagnetic units. The coordinates r, σ are real and ζ is complex with complex conjugate $\bar{\zeta}$.

The Weyl and Maxwell tensors have, in coordinates $x^i = (\zeta, \bar{\zeta}, r, \sigma)$,

$$k^i = \delta^i_3 = -g^{i4} \tag{2.6}$$

as a principal null direction (which is degenerate in the case of the Weyl tensor). In (2.6) g^{ij} is the inverse of the metric tensor g_{ij} given by (2.1). The integral curves of (2.6) are geodesic with affine parameter r along them. Thus

$$k^i{}_{|j}k^j = 0, \tag{2.7}$$

the stroke indicating covariant differentiation with respect to the metric g_{ij} given by

(2.1). In addition k^i is twist-free, shear-free and has expansion r^{-1}. The latter is expressed by the equation

$$\tfrac{1}{2}k^i{}_{|i} = r^{-1}. \tag{2.8}$$

The group of coordinate transformations which preserve the form of (2.1) and (2.2) is (cf Robinson and Trautman (1962))

$$\zeta = \psi(\zeta'), \qquad r = r'/\dot{\gamma}(\sigma'), \qquad \sigma = \gamma(\sigma'), \tag{2.9}$$

with ψ an analytic function of ζ' and the dot indicating differentiation with respect to σ'. Under (2.9) the quantities appearing in (2.1)–(2.3) transform in the following way:

$$\begin{aligned} m' &= \dot{\gamma}^3 m, \qquad K' = \dot{\gamma}^2 K, \qquad H' = \dot{\gamma}H + \ddot{\gamma}\dot{\gamma}^{-1}, \\ P' &= \dot{\gamma}P|\partial\psi/\partial\zeta'|^{-1}, \qquad e' = \dot{\gamma}^2 e, \qquad w' = \dot{\gamma}^{-1} w. \end{aligned} \tag{2.10}$$

The first four equations in (2.10) have been quoted in Robinson and Trautman (1962). We note that M, given by (2.3f), transforms as

$$M' = \dot{\gamma}^3 M, \tag{2.11}$$

in the same fashion as m in (2.10).

3. Energy–momentum tensor

Consider the two-index symmetric tensor T^{ij} given by

$$4\pi T^{ij} = r^{-2} J k^i k^j, \qquad J = J(\zeta, \bar{\zeta}, \sigma). \tag{3.1}$$

On account of (2.7) and (2.8) we have

$$T^{ij}{}_{|j} = 0, \tag{3.2}$$

irrespective of the choice of the function J. We shall tentatively refer to (3.1) as an energy–momentum tensor. A reasonable requirement of (3.1) is that it should be invariant under the group of transformations (2.9). From (2.6) and (2.1) we have $k_i \, dx^i = -d\sigma$ and thus under (2.9)

$$4\pi T_{ij} \, dx^i \, dx^j = r^{-2} J \, d\sigma^2 = r'^{-2} J' \, d\sigma'^2, \tag{3.3}$$

provided

$$J' = \dot{\gamma}^4 J. \tag{3.4}$$

Two obvious quantities which have this transformation property under (2.6), and have the required dimensions, i.e. are dimensionless in our units, are

$$\dot{M} - 3HM \qquad \text{and} \qquad e^2 N. \tag{3.5}$$

If we choose $J = e^2 N$ then T^{ij} in (3.1) becomes the radiative, i.e. $O(r^{-2})$, part of the electromagnetic energy–momentum tensor for the solutions described in § 1. We reject this choice because we would have $T^{ij} = 0$ when $e = 0$ and we wish to be able to calculate from (3.1) a non-zero rate of radiation of four-momentum in the neutral case $e = 0$. On the other hand the first expression of (3.5) is a serious contender. If $e = 0$ it becomes

$$\dot{m} - 3Hm. \tag{3.6}$$

Under (2.6) this transforms as in (3.4) and this has led Robinson and Trautman (1962) to remark that '$\dot{m}-3Hm \neq 0$ is an invariant statement, and it may be taken, very tentatively, as a criterion for radiation in the case $m \neq 0$'.

In the light of these observations, and rather than prolong the discussion, we propose to take

$$J = -(\dot{M} - 3HM). \tag{3.7}$$

This choice, including the choice of minus sign, will be justified at the level of the linearised theory in the following section.

4. Linearised theory

In the linearised theory of (2.1)–(2.4) the metric is expanded about the Minkowskian metric in the coordinates $(\zeta, \bar{\zeta}, r, \sigma)$ (cf Hogan and Imaeda 1979b). In the background Minkowskian space–time $r = 0$ is a timelike world-line (the history of the source) and σ is proper time along it. The null vector field in (2.6) is tangent to the generators of the future null-cones with vertices on $r = 0$. These null-cones have equations σ = constant. The connection between the coordinates $(\zeta, \bar{\zeta}, r, \sigma)$ and rectangular Cartesian coordinates and time $X^i = (x, y, z, t)$ is

$$X^i = x^i(\sigma) + rk^i, \tag{4.1}$$

where clearly $X^i = x^i(\sigma)$ is the equation of $r = 0$. In the coordinates X^i one has (Hogan and Imaeda 1979a)

$$k^i{}_{,j}k^j = 0, \tag{4.2}$$

the comma indicating partial differentiation with respect to X^j, and

$$\tfrac{1}{2}k^i{}_{,i} = r^{-1}. \tag{4.3}$$

Thus, when viewed in terms of X^i, T^{ij} given by (3.1) satisfies

$$T^{ij}{}_{,j} = 0. \tag{4.4}$$

Referring to Hogan and Imaeda (1979b), the linearised form of the expression (3.7) is

$$J = 3m\mu^i k_i - 2e^2\nu^i k_i - 8e^2(\mu^i k_i)^2. \tag{4.5}$$

Here m and e^2 are small of first order, μ^i is the four-acceleration of the source $r = 0$, $\nu^i = \mathrm{d}\mu^i/\mathrm{d}\sigma$ and we have chosen e = constant and m = constant. One can show that unless both e and m are chosen to be constants, the linearised field components will have unavoidable wire singularities.

If we apply the conservation law (4.4) to the four-volume of Minkowskian space–time bounded by the null-cones $\sigma = \sigma_1$, $\sigma = \sigma_2 (\sigma_2 > \sigma_1)$ and the three-surfaces $r = r_1$, $r = r_2 (r_2 > r_1)$ it is a straightforward matter, using the calculations of Synge (1970), to show that since T^{ij} is proportional to $k^i k^j$ there is no flux of four-momentum across the null-cones $\sigma = \sigma_1$, $\sigma = \sigma_2$ and that therefore the flux of four-momentum across $r = r_1$ must equal the flux of four-momentum across $r = r_2$. Thus the flux of four-momentum across a three-surface r = constant, calculated with T^{ij} given by (3.1) and (4.5), and thus satisfying (4.4), is independent of the particular three-surface r = constant, for $0 < r < +\infty$.

The rate of change with proper time σ of the outward flux of four-momentum across $r=$ constant is given by (cf Synge (1970))

$$\frac{dP^i}{d\sigma}=\int T^{ij}r_{,j}r^2\,d\omega, \tag{4.6}$$

where, using the formulae given in Hogan and Imaeda (1979a),

$$r_{,j}=-\lambda_j+Ck_j, \tag{4.7}$$

where $\lambda^j=dx^j/d\sigma(\lambda^j\lambda_j=-1)$ is the four-velocity of $r=0$ and

$$C=1+r\mu^ik_i. \tag{4.8}$$

Also k^i is normalised so that $k^i\lambda_i=-1$ and $d\omega$ is the surface element on the unit sphere. By (3.1), (4.5) and (4.7) we have

$$4\pi r^2T^{ij}r_{,j}=[3m\mu^jk_j-2e^2\nu^jk_j-8e^2(\mu^jk_j)^2]k^i. \tag{4.9}$$

We may write

$$k^i=\lambda^i+p^i, \qquad p^ip_i=+1, \qquad p^i\lambda_i=0, \tag{4.10}$$

and using

$$\mu^i\lambda_i=0, \qquad \nu^i\lambda_i=-\mu^i\mu_i=-\mu^2, \tag{4.11}$$

and the results

$$\int d\omega=4\pi, \qquad \int p^i\,d\omega=0, \qquad \int p^ip^j\,d\omega=\frac{4\pi}{3}(\eta^{ij}+\lambda^i\lambda^j), \qquad \int p^ip^jp^k\,d\omega=0, \tag{4.12}$$

a calculation of (4.6) yields

$$\frac{dP^i}{d\sigma}=-m\mu^i+\tfrac{2}{3}e^2\nu^i=-\frac{d}{d\sigma}(m\lambda^i-\tfrac{2}{3}e^2\mu^i). \tag{4.13}$$

From this we see that if $e=0$ then the rate of radiation of four-momentum P^i equals the rate of loss (hence the minus sign in (3.7)) of four-momentum $m\lambda^i$ of the source $r=0$. If the source is charged then the rate at which energy E is escaping is calculated from (4.13). Since it is a Lorentz invariant result we can obtain it by first choosing a frame of reference, at some $\sigma=$ constant, for which $\lambda^i=\delta^i_4$. Then, by (4.11), in this frame $\mu^4=0$, $\nu^4=\mu_\alpha\mu_\alpha(\alpha=1,2,3)$, and (4.13) gives

$$dP^4/d\sigma=\tfrac{2}{3}e^2\mu_\alpha\mu_\alpha. \tag{4.14}$$

This may be expressed in an arbitrary frame of reference as

$$dE/d\sigma=\tfrac{2}{3}e^2\mu^2, \tag{4.15}$$

where $E=-\lambda_iP^i$. We have obtained in (4.15) the well-known Larmor formula (cf Synge (1965) p 392).

If $e\neq0$ and we take its equation of motion to be the Lorentz–Dirac equation

$$m\mu^i=\tfrac{2}{3}e^2(\nu^i-\mu^2\lambda^i), \tag{4.16}$$

(we have already noted (Hogan and Imaeda 1979b) that such an equation emerges from the present theory without the occurrence of an infinite self-energy term) then (4.13) simplifies to

$$dP^i/d\sigma = \tfrac{2}{3}e^2\mu^2\lambda^i, \tag{4.17}$$

the known formula for the radiation rate of four-momentum by a run-away charge, calculated with the Liénard–Wiechert potential (see, for example, Schild (1960)).

Finally we observe that (4.13) suggests that we might refer to the quantity

$$p^i = m\lambda^i - \tfrac{2}{3}e^2\mu^i \tag{4.18}$$

as the four-momentum of a mass m of charge e.

5. Discussion

The extension of the calculation of (4.6) beyond the linear approximation would logically appear to involve the use of the form (3.2) of the conservation law rather than the Lorentz covariant form (4.4). One can, of course, calculate the O_2-contribution to J in (4.5) using (3.7), and use it directly in the evaluation of (4.6). This would appear to be an incorrect procedure. We have carried it out both for a uniformly accelerated source and the run-away charge. In the former case the O_2-contribution to (4.6) is divergent, while it is finite but ambiguous (involving an arbitrary function of σ arising from the integration of the field equations) in the latter case.

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